

# Maximum Likelihood Estimation of Signal-to-Noise Ratio and Combiner Weight

S. Kalson and S. J. Dolinar

Communications Systems Research Section

*An algorithm for estimating signal-to-noise ratio and combiner weight parameters for a discrete time series is presented. The algorithm is based upon the joint maximum likelihood estimate of the signal and noise power. The discrete-time series are the sufficient statistics obtained after matched filtering of a biphase modulated signal in additive white gaussian noise, before maximum likelihood decoding is performed.*

## I. Introduction and Problem Model

This article investigates maximum likelihood estimation of signal-to-noise ratio and combiner weight parameters for a discrete time series. The discrete time series are the sufficient statistics obtained after matched filtering of a biphase modulated signal (Ref. 1). In order to show the underlying assumptions and limitations of the estimation problem, we first examine the communication system that gives rise to the discrete time series.

We take as our model that given in Fig. 1. The channel encoder maps the binary digital source encoder output  $\{I_k\}$  into the binary channel symbols  $\{C_k\}$ , where the channel symbols are produced with rate  $1/T$ . The modulation is biphase. That is, the modulator produces the baseband signal

$$s(t) = \sum_k A_k q_k(t) \quad (1)$$

where the  $\{A_k\}$  are chosen according to

$$A_k = \begin{cases} -\sqrt{E_s}, & C_k = \text{"0"} \\ +\sqrt{E_s}, & C_k = \text{"1"} \end{cases} \quad (2)$$

Here,  $E_s$  is the channel symbol energy, and the  $\{q_k(t)\}$  are orthonormal basis functions. We assume that the  $\{q_k(t)\}$  are time-displaced replicas of a single function of duration  $T$ , namely,

$$q_k(t) = q(t - (k-1)T) \quad (3)$$

where

$$q(t) = 0, \quad t < 0 \text{ or } t > T \quad (4)$$

$$\int_0^T q(t)^2 dt = 1 \quad (5)$$

The baseband signal  $s(t)$  is transmitted over an additive white gaussian noise channel with one-sided noise spectral

density  $N_0$ . The received baseband signal is represented by  $x(t)$  in Fig. 1. This received signal is demodulated by matched filtering (integrate and dump) to produce a discrete time series  $\{x_k\}$ :

$$x_k = \int_{(k-1)T}^{kT} x(t)q(t - (k-1)T) dt \quad (6)$$

Assuming such ideal channel and receiver characteristics as perfect phase tracking and channel symbol synchronization, no channel symbol interference, etc., the output time series from the demodulator are the sufficient statistics for maximum likelihood decoding. Referring to Eqs. (1)–(6), we see that this time series can be written in the form

$$x_k = m_{\text{true}} a_k + \sigma_{\text{true}} b_k \quad (7)$$

where the  $\{a_k\}$  are either +1 or -1 depending upon whether the channel symbol transmitted was a “1” or “0,”  $\{b_k\}$  are independent and identically distributed gaussian random variables with zero mean and unit variance, and  $m_{\text{true}}$  and  $\sigma_{\text{true}}$  are given by

$$m_{\text{true}} = \sqrt{E_s} \quad (8)$$

$$\sigma_{\text{true}} = \sqrt{N_0/2} \quad (9)$$

The parameters  $m_{\text{true}}$  and  $\sigma_{\text{true}}$  represent the true values of the signal and noise amplitudes, respectively. We note that  $\sigma_{\text{true}}$  and  $m_{\text{true}}$  are by definition non-negative.

In order to make the problem mathematically tractable, we make the assumption that the  $\{a_k\}$  are independent and take on the values +1 and -1 with equal probability. For a communication system employing coding, this assumption is not correct. Thus, the effect of coding on the estimation algorithm given here needs to be determined.

## II. Parameters to be Estimated

Our starting point for all further analysis is the time series  $\{x_k\}$  defined in Eq. (7), with the assumed probabilistic models for the sequence of random variables  $\{a_k\}$  and  $\{b_k\}$ . Our objective is to find maximum likelihood estimates of the signal and noise parameters  $m_{\text{true}}$  and  $\sigma_{\text{true}}$  or of other parameters of interest that are embedded in the model. Two such parameters are signal-to-noise ratio and combiner weight.

The signal-to-noise ratio (SNR) at the receiver is defined as

$$\text{SNR} = E_s/N_0 \quad (10)$$

SNR is a fundamental parameter of interest for a variety of reasons. For example, SNR is needed to optimally choose the quantization levels of the demodulator so that the “best” discrete channel is provided to the channel encoder-decoder (Ref. 2). We find it convenient to define a signal-to-noise ratio parameter  $\rho_{\text{true}}$  for the demodulated time series as

$$\rho_{\text{true}} = m_{\text{true}}^2/\sigma_{\text{true}}^2 \quad (11)$$

In terms of  $\rho_{\text{true}}$ , the SNR at the receiver is simply

$$\text{SNR} = \rho_{\text{true}}/2 \quad (12)$$

Another quantity of interest is the combiner weight needed for symbol stream combining. For example, suppose  $L$  different time series (or symbol streams) are available from  $L$  different receiver-demodulators,

$$x_{ik} = m_i a_k + \sigma_i b_{ik}, \quad i = 1, 2, \dots, L \quad (13)$$

where, as before,  $\{a_k\}$  are either +1 or -1, and  $\{b_{ik}\}$  are independent, identically distributed gaussian random variables with zero mean and unit variance. It can be shown (Ref. 3) that maximum likelihood decoding of the  $L$  time series  $\{x_{ik}\}$  is equivalent to maximum likelihood decoding of a single time series  $\{y_k\}$ , where

$$y_k = \sum_{i=1}^L \alpha_i x_{ik} \quad (14)$$

and the combiner weights  $\{\alpha_i\}$  are chosen to be proportional to  $\{m_i/\sigma_i^2\}$ . Thus, we are interested in estimating for any given time series a combiner weight parameter defined by

$$\alpha_{\text{true}} = m_{\text{true}}/\sigma_{\text{true}}^2 \quad (15)$$

In different applications, we may desire to estimate one, two, or several parameters simultaneously. However, we should always be aware that our assumed problem model has exactly two independent unknown parameters. This implies that any estimate of a single parameter (such as SNR) must be aided by an implicit estimate of an independent auxiliary parameter, and that simultaneous estimates of more than two parameters are not all independent. In particular, maximum likelihood estimation as applied to our problem must produce a joint maximum likelihood estimate of a *pair* of independent parameters.

Fortunately, it is not necessary to re-solve the maximum likelihood equations for every combination of parameters of interest. If two pairs of parameters are related by a one-to-one

transformation, then the corresponding joint maximum likelihood estimates are related by the same transformation (Ref. 4). Thus, we propose finding the joint maximum likelihood estimate of the signal and noise parameters  $m_{\text{true}}$  and  $\sigma_{\text{true}}$ , which we denote as  $\hat{m}$  and  $\hat{\sigma}$ , respectively. Then (ignoring a singularity at  $\sigma_{\text{true}} = 0$  or  $\hat{\sigma} = 0$ ) we can define the corresponding maximum likelihood estimates of the signal-to-noise ratio parameter  $\rho_{\text{true}}$  and the combiner weight parameter  $\alpha_{\text{true}}$  as

$$\hat{\rho} = \hat{m}^2 / \hat{\sigma}^2 \quad (16)$$

$$\hat{\alpha} = \hat{m} / \hat{\sigma}^2 \quad (17)$$

### III. The Log-Likelihood Function

Let us denote a set of  $N$  measurements  $(x_1, x_2, \dots, x_N)$  by the vector  $\mathbf{x}$ . The probability density function of  $\mathbf{x}$  conditioned on  $m_{\text{true}} = m$  and  $\sigma_{\text{true}} = \sigma$  is

$$p(\mathbf{x}|m, \sigma) = \prod_{k=1}^N \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \left\{ \exp \frac{-(x_k - m)^2}{2\sigma^2} + \exp \frac{-(x_k + m)^2}{2\sigma^2} \right\} \quad (18)$$

which after a little algebra becomes

$$p(\mathbf{x}|m, \sigma) = \prod_{k=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-x_k^2}{2\sigma^2} \exp \frac{-m^2}{2\sigma^2} \cosh \frac{mx_k}{\sigma^2} \quad (19)$$

Taking the natural logarithm of both sides of Eq. (19) gives the log-likelihood function:

$$\begin{aligned} \ln p(\mathbf{x}|m, \sigma) = & -N \ln \sqrt{2\pi} - N \ln \sigma - \frac{1}{2\sigma^2} \sum_{k=1}^N x_k^2 \\ & - \frac{Nm^2}{2\sigma^2} + \sum_{k=1}^N \ln \cosh \frac{mx_k}{\sigma^2} \end{aligned} \quad (20)$$

### IV. The Set of Feasible Solutions

Consider the  $\sigma - m$  plane where  $\sigma$  is the abscissa and  $m$  is the ordinate. The joint maximum likelihood estimate (MLE)

of  $\sigma_{\text{true}}$  and  $m_{\text{true}}$  is the ordered pair  $(\hat{\sigma}, \hat{m})$  in the  $\sigma - m$  plane where  $\ln p(\mathbf{x}|m, \sigma)$  obtains its maximum. Let us define a set of feasible solutions to the MLE problem as a set of ordered pairs in the  $\sigma - m$  plane of which the MLE  $(\hat{\sigma}, \hat{m})$  is a member. We wish to find a set of feasible solutions that is as small as possible. Since  $\sigma_{\text{true}}$  and  $m_{\text{true}}$  are non-negative, we can restrict the set of feasible solutions to lie in the first quadrant, including the non-negative  $\sigma$  and  $m$  axes.

A necessary condition for a function to obtain its maximum at some point in the *interior* of a closed, bounded region is that its partial derivatives at that point are zero. Although the first quadrant of the  $\sigma - m$  plane is not bounded, one can observe from Eq. (18) that for finite  $\{x_k\}$ ,  $p(\mathbf{x}|m, \sigma)$  approaches zero for large  $\sigma$  and  $m$ . Thus, the maximum of  $\ln p(\mathbf{x}|m, \sigma)$  must be contained in some bounded region. Therefore, we include in our set of feasible solutions those points in the first quadrant (excluding the non-negative axes) at which both partial derivatives of  $\ln p(\mathbf{x}|m, \sigma)$  with respect to  $\sigma$  and  $m$  vanish.

We must separately consider if the maximum might occur on the non-negative axes. Thus, a set of feasible solutions consists of those points in the first quadrant of the  $\sigma - m$  plane where both partial derivatives of  $\ln p(\mathbf{x}|m, \sigma)$  vanish, and those points on the non-negative axes where  $\ln p(\mathbf{x}|m, \sigma)$  obtains a local maximum. Let us first consider the latter.

#### A. $m$ -Axis Solutions

In the limit as  $\sigma \rightarrow 0$  ( $m$ -axis), we see from Eq. (18) that  $p(\mathbf{x}|m, \sigma)$  is proportional to the product of delta functions given below:

$$\lim_{\sigma \rightarrow 0} p(\mathbf{x}|m, \sigma) \sim \prod_{k=1}^N \{\delta(x_k - m) + \delta(x_k + m)\} \quad (21)$$

In this case, one can see that if there exists some constant  $c$  such that  $|x_k| = c$  for all  $k$ , then  $p(\mathbf{x}|m, \sigma)$  is zero everywhere on the  $m$ -axis except at  $m = c$ , where it is unbounded. Conversely, if the  $\{x_k\}$  are not all equal in magnitude, then  $p(\mathbf{x}|m, \sigma)$  is zero on the entire  $m$ -axis. Thus, since  $p(\mathbf{x}|m, \sigma)$  is bounded everywhere except possibly the  $m$ -axis, we can state that

$$(\hat{\sigma}, \hat{m}) = (0, c) \quad (22)$$

if and only if there exists a  $c$  such that  $|x_k| = c$  for all  $k$ .

#### B. $\sigma$ -Axis Solution

For  $m = 0$  ( $\sigma$ -axis), we have from Eq. (18) that

$$p(\mathbf{x}|m, \sigma) = \prod_{k=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp \frac{-x_k^2}{2\sigma^2} \quad (23)$$

This is just a unimodal gaussian density function with mean zero. It is well known (Ref. 4) that this function obtains its maximum at

$$\sigma = \sqrt{\frac{1}{N} \sum_{k=1}^N x_k^2}$$

Thus, the only point on the  $\sigma$ -axis that we need to include in the set of feasible solutions is

$$\left( \sqrt{\frac{1}{N} \sum_{k=1}^N x_k^2}, 0 \right) \quad (24)$$

### C. Interior First Quadrant Solutions

The other members of the set of feasible solutions are the points in the first quadrant of the  $\sigma - m$  plane (excluding the non-negative axes) where the partial derivatives of  $\ln p(\mathbf{x}|m, \sigma)$  with respect to  $\sigma$  and  $m$  vanish. Thus, we must find those ordered pairs  $(\sigma, m)$  for which both  $\sigma$  and  $m$  are positive and simultaneously satisfy

$$\frac{\partial}{\partial \sigma} \ln p(\mathbf{x}|m, \sigma) = 0 \quad (25)$$

and

$$\frac{\partial}{\partial m} \ln p(\mathbf{x}|m, \sigma) = 0 \quad (26)$$

Performing the indicated partial derivatives on Eq. (20) leads to

$$\begin{aligned} \frac{\partial}{\partial \sigma} \ln p(\mathbf{x}|m, \sigma) &= -\frac{N}{\sigma} + \frac{N}{\sigma^3} m^2 + \frac{1}{\sigma^3} \sum_{k=1}^N x_k^2 \\ &\quad - \frac{2m}{\sigma^3} \sum_{k=1}^N x_k \tanh \frac{mx_k}{\sigma^2} \end{aligned} \quad (27)$$

$$\frac{\partial}{\partial m} \ln p(\mathbf{x}|m, \sigma) = \frac{-Nm}{\sigma^2} + \frac{1}{\sigma^2} \sum_{k=1}^N x_k \tanh \frac{mx_k}{\sigma^2} \quad (28)$$

Setting Eq. (28) equal to zero leads to the relation between  $m$  and  $\sigma$ :

$$m = \frac{1}{N} \sum_{k=1}^N x_k \tanh \frac{mx_k}{\sigma^2} \quad (29)$$

Using Eq. (29), we can simplify the rightmost term in Eq. (27). Consequently, setting Eq. (27) equal to zero leads to the second relation between  $m$  and  $\sigma$ :

$$\sigma^2 + m^2 = \frac{1}{N} \sum_{k=1}^N x_k^2 \quad (30)$$

For simplicity of notation, let us make the definition

$$\langle x^2 \rangle_N \triangleq \frac{1}{N} \sum_{k=1}^N x_k^2 \quad (31)$$

For now, since we are only considering positive  $\sigma$  and  $m$ , we see from Eq. (30) that the feasible solutions  $(\sigma, m)$  in the first quadrant (excluding the non-negative axes) must satisfy  $0 < \sigma < \sqrt{\langle x^2 \rangle_N}$  and  $0 < m < \sqrt{\langle x^2 \rangle_N}$ . Using Eq. (30) to solve for  $\sigma$  in terms of  $m$  and substituting into Eq. (19), we obtain a transcendental equation in one unknown:

$$m = \frac{1}{N} \sum_{k=1}^N x_k \tanh \frac{mx_k}{\langle x^2 \rangle_N - m^2}, \quad 0 < m < \sqrt{\langle x^2 \rangle_N} \quad (32)$$

Thus, given the measurements  $x_k, k = 1, 2, \dots, N$ , a set of feasible solutions in the interior first quadrant consists of ordered pairs of the form  $(\sqrt{\langle x^2 \rangle_N - m^2}, m)$ , where  $m$  satisfies Eq. (32). Equivalently,  $m$  is one of the roots of the function  $F(m, \mathbf{x}) = m - f(m, \mathbf{x})$ , where

$$f(m, \mathbf{x}) \triangleq \frac{1}{N} \sum_{k=1}^N x_k \tanh \frac{mx_k}{\langle x^2 \rangle_N - m^2} \quad (33)$$

### V. Finding the Roots of $F(m, \mathbf{x})$

Rather than finding the roots of  $F(m, \mathbf{x})$  in the range  $0 < m < \sqrt{\langle x^2 \rangle_N}$ , let us extend this range to  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$ . At first, one may think that we have needlessly increased the size of the set of feasible solutions defined in the last section. However, we will see in this section that finding the roots of  $F(m, \mathbf{x})$  in this new range of  $m$  includes the feasible solutions on the  $\sigma$  and  $m$  axes.

Finding the roots of  $F(m, \mathbf{x})$  can be tricky. For example, looking for roots by investigating when  $F(m, \mathbf{x})$  changes sign may fail since  $F(m, \mathbf{x})$  may contain two or more roots very close together, or may in fact not change sign at a root. However, insight can be gained by observing that the roots of  $F(m, \mathbf{x})$  are just the intersection of the curves:

$$z = m \quad (34)$$

and

$$z = f(m, \mathbf{x}) \quad (35)$$

where  $m$  is restricted to  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$ .

It is interesting to note that  $f(m, \mathbf{x})$  is an even function in  $x_k$ . That is,  $f(m, \mathbf{x})$  depends on each  $x_k$  via its absolute value. This is not surprising since one can see that the conditional probability density function  $p(\mathbf{x}|m, \sigma)$  in Eq. (18) depends only on  $|x_k|$ . Thus, the absolute values of  $\{x_k\}$  constitute a sufficient statistic and the sign bit of  $x_k$  is not needed. Important properties of  $f(m, \mathbf{x})$  are listed in Table 1, where for notational convenience we have made the definitions:

$$\langle |x| \rangle_N = \frac{1}{N} \sum_{k=1}^N |x_k| \quad (36)$$

$$\langle x^4 \rangle_N \triangleq \frac{1}{N} \sum_{k=1}^N x_k^4 \quad (37)$$

We shall now show that the roots of  $F(m, \mathbf{x})$  in the range  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$  include the feasible solutions on the  $m$  and  $\sigma$  axes, as given in Eqs. (22) and (24). First, we verify Eq. (24), which specifies the feasible solution on the  $\sigma$ -axis. We see from Table 1, property (i), that  $m = 0$  is always a root of  $F(m, \mathbf{x})$  for all  $\mathbf{x}$ . But when  $m = 0$ , we have from Eq. (30) that  $\sigma = \sqrt{\langle x^2 \rangle_N}$ . Thus, the feasible solution  $(\sqrt{\langle x^2 \rangle_N}, 0)$  on the  $\sigma$ -axis can be obtained from finding the roots of  $F(m, \mathbf{x})$  in the range  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$ .

Next, we show that finding the roots of  $F(m, \mathbf{x})$  within the range  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$  also specifies the feasible solution on the  $m$ -axis as given by Eq. (22). It is not too difficult to see from Eq. (22) that  $(0, \sqrt{\langle x^2 \rangle_N})$  is the joint MLE if and only if there exists a  $c$  such that  $|x_k| = c$  for all  $k$ . However, it is easily verified that  $|x_k| = c$  for all  $k$  implies;

$$\sqrt{\langle x^2 \rangle_N} = \langle |x| \rangle_N \quad (38)$$

in which case we have using property (ii) of Table 1:

$$f(\sqrt{\langle x^2 \rangle_N}, \mathbf{x}) = \langle |x| \rangle_N = \sqrt{\langle x^2 \rangle_N} \quad (39)$$

Thus,  $m = \sqrt{\langle x^2 \rangle_N}$  is a root of  $F(m, \mathbf{x})$  whenever  $|x_k| = c$  for all  $k$ . Furthermore, for  $m = \sqrt{\langle x^2 \rangle_N}$ , we have from Eq. (30) that  $\sigma = 0$ . It thus follows that whenever the ordered pair  $(0, \sqrt{\langle x^2 \rangle_N})$  is the joint MLE, it can always be obtained by looking for the roots of  $F(m, \mathbf{x})$  in the range  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$ .

Having justified extending the search for the roots of  $F(m, \mathbf{x})$  to the range  $0 \leq m \leq \sqrt{\langle x^2 \rangle_N}$ , let us state what we currently know regarding the roots within this range. As mentioned before,  $m = 0$  is always a root of  $F(m, \mathbf{x})$ . Are there any nonzero roots? To answer this question, we first note that the curve  $z = f(m, \mathbf{x})$  is not above the curve  $z = m$  at  $m = \sqrt{\langle x^2 \rangle_N}$ . This is easily verified by invoking Jensen's inequality

$$\langle |x| \rangle_N \leq \sqrt{\langle x^2 \rangle_N} \quad (40)$$

and using property (ii) of Table 1 to yield

$$f(\sqrt{\langle x^2 \rangle_N}, \mathbf{x}) = \langle |x| \rangle_N \leq \sqrt{\langle x^2 \rangle_N} \quad (41)$$

Next, we observe that from property (vi) of Table 1, the curve  $z = f(m, \mathbf{x})$  rises above the curve  $z = m$  sufficiently near  $m = 0$  if and only if the following critical condition is satisfied:

$$\langle x^4 \rangle_N < 3 \langle x^2 \rangle_N^2 \quad (42)$$

Thus, if Eq. (42) is satisfied, the curve  $z = f(m, \mathbf{x})$  must intersect the curve  $z = m$  for some nonzero  $m$  less than or equal to  $\sqrt{\langle x^2 \rangle_N}$ .

The condition in Eq. (42) is interesting because it parallels an easily verifiable relationship between the corresponding ensemble averages, namely,  $E\{x^4\} < 3 E\{x^2\}^2$  for  $m_{\text{true}} > 0$ , and  $E\{x^4\} = 3 E\{x^2\}^2$  for  $m_{\text{true}} = 0$ . Thus, a nonzero root of  $F(m, \mathbf{x})$  is guaranteed whenever the sample moments  $\langle x^4 \rangle_N, \langle x^2 \rangle_N$  bear the same relationship as that relationship between ensemble moments which distinguishes the nonzero-mean case from the zero-mean case.

Finally, we state one more property that is known concerning the roots of  $F(m, \mathbf{x})$ . As mentioned before,  $|x_k| = c$  for all  $k$  implies that  $m = \sqrt{\langle x^2 \rangle_N}$  is a root of  $F(m, \mathbf{x})$ . The converse is also true. If  $\sqrt{\langle x^2 \rangle_N}$  is a root of  $F(m, \mathbf{x})$ , then

$$f(\sqrt{\langle x^2 \rangle_N}, \mathbf{x}) = \sqrt{\langle x^2 \rangle_N} \quad (43)$$

and thus from property (ii) we have

$$\langle |x| \rangle_N = \sqrt{\langle x^2 \rangle_N} \quad (44)$$

which implies  $|x_k| = c$  for all  $k$ , because this is the only condition under which Jensen's inequality, Eq. (40), can be satisfied with equality.

We summarize our results in this section in the following theorem:

### Theorem 1

- (1) The feasible solutions are of the form  $(\sqrt{\langle x^2 \rangle_N - m^2}, m)$ , where  $m$  is a root of  $F(m, \mathbf{x})$  in the range  $[0, \sqrt{\langle x^2 \rangle_N}]$ .
- (2) (a)  $F(m, \mathbf{x})$  always has the root  $m = 0$  for all  $\mathbf{x}$ .  
 (b)  $F(m, \mathbf{x})$  has the root  $m = \sqrt{\langle x^2 \rangle_N}$  if and only if  $|x_k| = \sqrt{\langle x^2 \rangle_N}$  for all  $k$ .  
 (c) If  $\langle x^4 \rangle_N < 3 \langle x^2 \rangle_N^2$ , then there exists a non-zero root of  $F(m, \mathbf{x})$  less than or equal to  $\sqrt{\langle x^2 \rangle_N}$ .

In Fig. 2, we have sketched  $z = m$  and a hypothetical  $z = f(m, \mathbf{x})$  satisfying Eq. (42). For sake of simplicity, we have drawn the curves so that there are only two intersections.

Unfortunately, Theorem 1 is all that we know regarding  $F(m, \mathbf{x})$ . Several pertinent questions are: If Eq. (42) is satisfied, is there only one non-zero root? If Eq. (42) is not satisfied, are there any non-zero roots? And, finally, when there is more than one root, which one corresponds to the MLE? These questions have been very difficult to answer analytically. It would be "nice" if there were only one non-zero root when Eq. (42) is satisfied, and no non-zero roots otherwise. A few plots of  $F(m, \mathbf{x})$  indicate that this might be so. Properties of  $F(m, \mathbf{x})$  which might give some indication about the number of roots are currently being investigated. We are also in the process of looking for counterexamples.

## VI. An Algorithm for an Upper Bound of the MLE

Although Theorem 1 is somewhat incomplete concerning the number of roots of  $F(m, \mathbf{x})$ , we can nevertheless give an algorithm for finding the largest root, which provides an upper bound to our signal-to-noise ratio and combiner weight estimators. We suspect that this upper bound is indeed the MLE and we show later that this is true in the large SNR case.

Let  $m^*$  denote the largest root of  $F(m, \mathbf{x})$ , where  $0 \leq m^* \leq \sqrt{\langle x^2 \rangle_N}$ . A graphical representation of the algorithm for finding  $m^*$  is given in Fig. 3. At the  $i$ th iteration,  $m^{(i)}$  is some estimate of  $m^*$ , where  $m^{(i)} \geq m^*$ . As the figure indicates, the better estimate  $m^{(i+1)}$  is obtained by following the paths labeled (1), (2), and (3). One can see that the estimate  $m^{(i+1)}$  is closer to  $m^*$  than the previous estimate  $m^{(i)}$ , but is still larger than  $m^*$ . From Fig. 3, we see that  $m^{(i+1)}$  is given simply by the recursion

$$m^{(i+1)} = f(m^{(i)}, \mathbf{x}) \quad (45)$$

The zeroth estimate of  $m^*$  is

$$m^{(0)} = \sqrt{\langle x^2 \rangle_N}$$

Thus, performing Eq. (45) for  $i = 0, 1, \dots$ , generates the sequence of estimates  $m^{(1)} > m^{(2)} > \dots$ . It can be shown that this sequence converges to  $m^*$ . Since  $m^*$  is the largest root of  $F(m, \mathbf{x})$ , we see that upper bounds to our estimators are

$$\hat{\rho} = \frac{\hat{m}^2}{\hat{\sigma}^2} \leq \lim_{i \rightarrow \infty} \frac{[m^{(i)}]^2}{\langle x^2 \rangle_N - [m^{(i)}]^2} \quad (46)$$

$$\hat{\alpha} = \frac{\hat{m}}{\hat{\sigma}} \leq \lim_{i \rightarrow \infty} \frac{m^{(i)}}{\langle x^2 \rangle_N - [m^{(i)}]^2} \quad (47)$$

To obtain a qualitative understanding of how the rate of convergence of such an algorithm depends on SNR, let us make the definition:

$$\Delta^{(i)} \triangleq m^{(i)} - m^* \quad (48)$$

By the mean value theorem, there exists some  $m_0$  between  $m^*$  and  $m^{(i)}$  such that

$$\Delta^{(i+1)} = m^{(i+1)} - m^* = f(m^{(i)}, \mathbf{x}) - f(m^*, \mathbf{x}) = \left. \frac{\partial f}{\partial m} \right|_{m=m_0} \Delta^{(i)} \quad (49)$$

Equation (49) gives us some idea about the rate of convergence of the algorithm. For example, in the case of high SNR,  $m^*$  should be close to  $\sqrt{\langle x^2 \rangle_N}$  and in this case the partial derivative of  $f(m, \mathbf{x})$  should be close to zero. Thus, one can see from Eq. (49) that  $\Delta^{(i)}$  would approach zero very rapidly. On the other hand, for low SNR, one would expect that  $m^*$  would be closer to zero, and consequently the partial derivative of  $f(m, \mathbf{x})$  would be closer to its derivative at the origin, which is one. In this case the convergence would be very slow.

An algorithm for finding a root close to  $m^*$  is given by the flow diagram in Fig. 4. The variable TOL is a pre-assigned tolerance for the difference between two successive estimates of  $m^*$ . Also, a limit to the number of iterations of the algorithm is set by the variable NUM. This is needed in the low SNR case where the convergence of the estimate to  $m^*$  may be asymptotically slow.

## VII. The High SNR Case

It is interesting to consider the high SNR case, especially since this serves as a check on our method. In the high SNR case, the  $\{x_k\}$  will most likely be nearly equal in magnitude. Then, from Jensen's inequality, Eq. (40),  $\langle |x| \rangle_N$  would be close to, but less than  $\sqrt{\langle x^2 \rangle_N}$ . Thus, from Fig. 3, we expect the intersection of the curves  $z = m$  and  $z = f(m, x)$  to be close to  $\sqrt{\langle x^2 \rangle_N}$ , in which case just one iteration of the algorithm of Fig. 4 would yield a close estimate of  $m^*$ , given below.

$$m^* \approx m^{(1)} = f(\sqrt{\langle x^2 \rangle_N}, x) = \langle |x| \rangle_N, \text{ for SNR} \rightarrow \infty \quad (50)$$

If we use the above for  $m^*$ , then the signal-to-noise ratio estimate is

$$\hat{\rho} \approx \frac{\langle |x|^2 \rangle_N}{\langle x^2 \rangle_N - \langle |x| \rangle_N^2} \quad (51)$$

which asymptotically equals the usual signal-to-noise ratio estimate for the high SNR case (Ref. 5).

## VIII. Summary

The main result of this memo is an algorithm for finding upper bounds to the maximum likelihood estimates of signal-to-noise ratio and combiner weights. Further work is needed to determine if these upper bounds equal the maximum likelihood estimates.

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**Table 1. Properties of  $f(m, x)$**

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$$f(0, x) = 0 \quad (\text{i})$$

$$\lim_{m \rightarrow \sqrt{\langle x^2 \rangle_N}} f(m, x) = \langle |x| \rangle_N \quad (\text{ii})$$

$$\left. \frac{\partial f}{\partial m} \right|_{m=0} = 1 \quad (\text{iii})$$

$$\left. \frac{\partial f}{\partial m} \right|_{m \rightarrow \sqrt{\langle x^2 \rangle_N}} = 0 \quad (\text{iv})$$

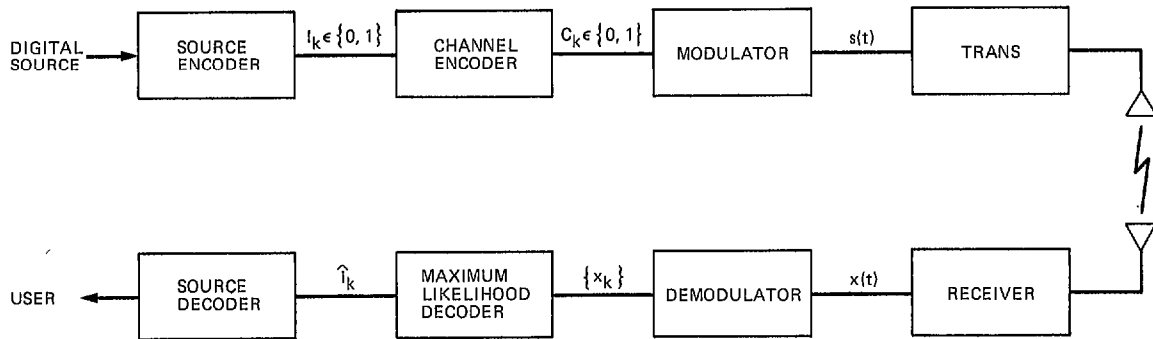
$$f(m, x) \text{ is monotonically increasing in } m \quad (\text{v})$$

To third order in  $m$ :

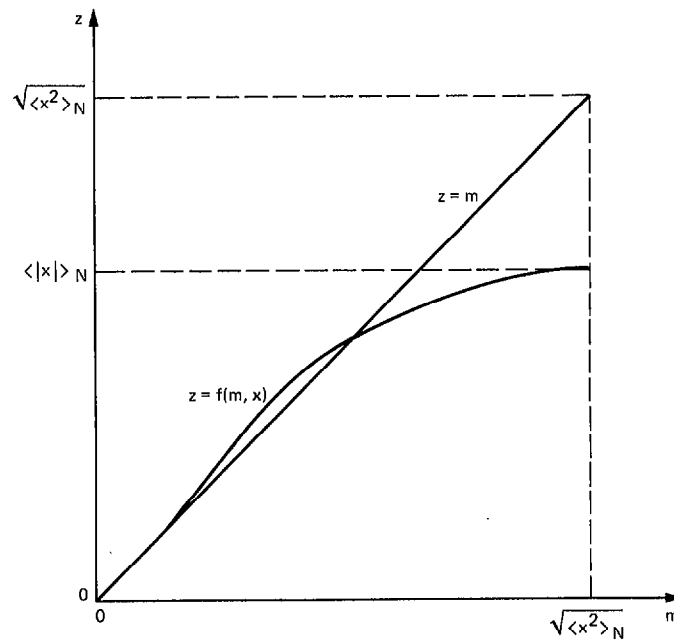
$$f(m, x) \sim m + \frac{m^3}{\langle x^2 \rangle_N} \left[ 1 - \frac{\langle x^4 \rangle_N}{3 \langle x^2 \rangle_N^2} \right] \quad (\text{vi})$$


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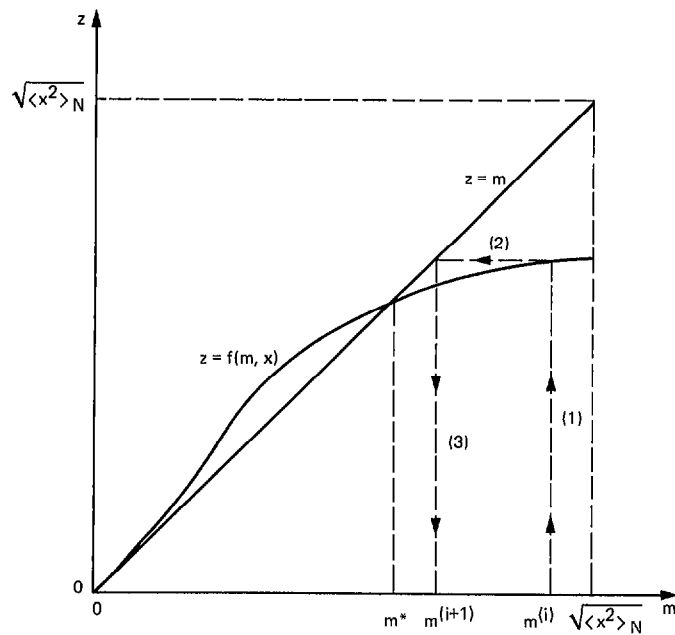




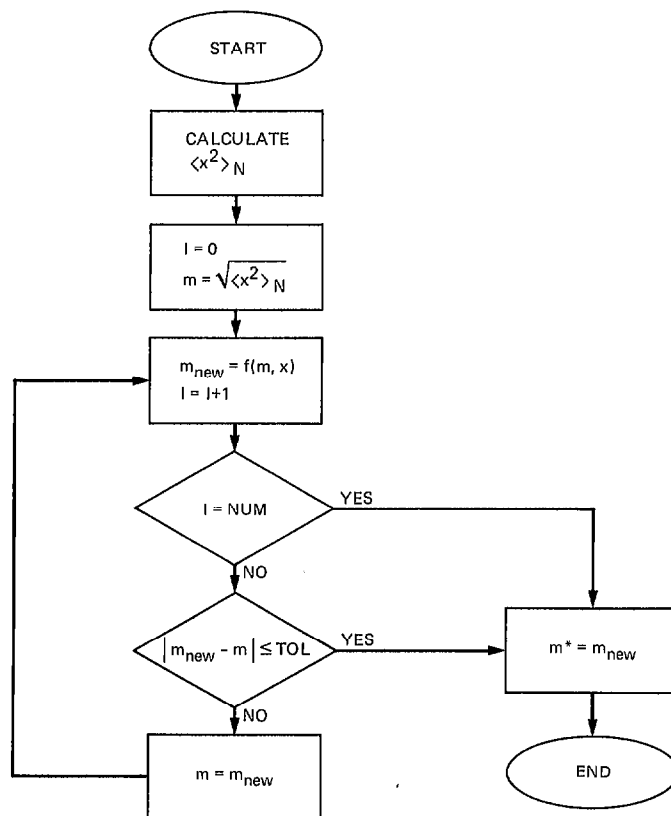
**Fig. 1. Model of communication system**



**Fig. 2. Sketch of  $z = m$  and  $z = f(m, x)$**



**Fig. 3. Graphical representation of an iterative procedure to find  $m^*$**



**Fig. 4. Algorithm for finding  $m^*$**